Exercise 53

Estimate the horizontal asymptote of the function

$$f(x) = \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000}$$

by graphing f for $-10 \leq x \leq 10$. Then calculate the equation of the asymptote by evaluating the limit. How do you explain the discrepancy?

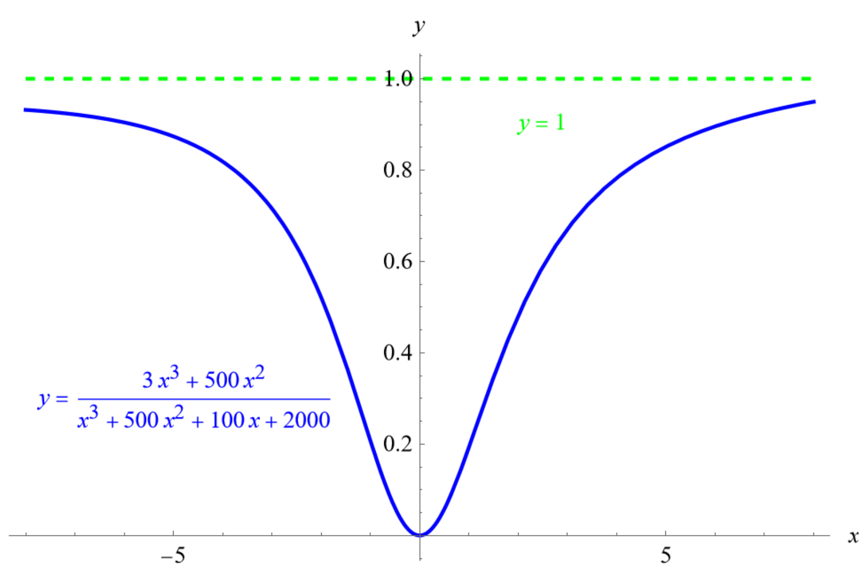
Solution

Calculate the limits as $x \to \pm\infty$ to determine the horizontal asymptote. In the second limit, make the substitution, $x = -u$, so that as $x \to -\infty$, $u \to \infty$.

$$\lim_{x\to\infty} \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000} = \lim_{x\to\infty} \frac{3 + \frac{500}{x}}{1 + \frac{500}{x} + \frac{100}{x^2} + \frac{2000}{x^3}} = \frac{3+0}{1+0+0} = 3$$

$$\begin{aligned}
\lim_{x\to-\infty} \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000} &= \lim_{u\to\infty} \frac{3(-u)^3 + 500(-u)^2}{(-u)^3 + 500(-u)^2 + 100(-u) + 2000} \\
&= \lim_{u\to\infty} \frac{-3u^3 + 500u^2}{-u^3 + 500u^2 - 100u + 2000} \\
&= \lim_{u\to\infty} \frac{-3 + \frac{500}{u}}{-1 + \frac{500}{u} - \frac{100}{u^2} + \frac{2000}{u^3}} \\
&= \frac{-3+0}{-1+0-0+0} \\
&= 3
\end{aligned}$$

Therefore, the horizontal asymptote is $y = 3$. Notice in the graphs below how the horizontal asymptotes can be tricky to find.



Zoom out enough to find them.